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Cosmological ϕ^4, ϕ^6, and Sine-Gordon Theories with Broken Symmetry

Huang Baofa[1] **and Wang Jingchang**[1]

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Particular solutions to the Einstein equations with cosmological constant are presented and discussed for an isotropic, spatially homogeneous, and spatially flat space-time in the case that the matter fields are ϕ^4, ϕ^6, and sine-Gordon fields.

1. INTRODUCTION

The study of quantum field theory in curved space-time has become increasingly relevant since the introduction of the inflationary universe scenarios. The theory is normally formulated in flat space-time, but the behavior in a curved space-time may be vastly different in several important aspects. In this paper, we describe a particular solution to the Einstein equations for a spatially flat ($k=0$) Robertson–Walker space-time. The matter content is a real scalar field, possessed of a self-interaction according to the Lagrangian (Linde, 1979)

$$\mathscr{L}=\frac{1}{2}\,\partial^u\phi\,\partial_u\phi-V(\phi) \tag{1.1}$$

The metric is

$$g_{uv}\,dx^u\,dx^v=dt^2-R^2(t)(dx^2+dy^2+dz^2) \tag{1.2}$$

The Euler–Lagrange equation obtained by varying the action S

$$S=\int dx^4\,\sqrt{-g}\,\mathscr{L} \tag{1.3}$$

is

$$\Box\phi+\dot{V}(\phi)=0 \tag{1.4}$$

[1]Shanghai Technical College of Metallurgy, Shanghai 200232, China.

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Here overdots denote a partial derivative with respect to time,

$$\Box\phi \equiv g^{uv}\phi_{;uv}, \qquad g \equiv \det(g_{uv}) \tag{1.5}$$

A semicolon denotes a covariant derivative.

If the scalar field is required to share the symmetry of the space-time, then $\phi = \phi(t)$ and (1.4) becomes (with $\dot{\phi} = d\phi/dt$)

$$\ddot{\phi} + 3\frac{\dot{R}}{R}\dot{\phi} + \dot{v}(\phi) = 0 \tag{1.6}$$

The energy-momentum tensor is obtained by functional differentiation of the action (1.3) with respect to the metric tensor (Birrell and Davies, 1982), that is,

$$T_{uv} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{uv}} \tag{1.7}$$

which yields

$$T_{uv} = \partial_u\phi\,\partial_v\phi - \mathscr{L}g_{uv} \tag{1.8}$$

The Einstein equations

$$R_{uv} + \tfrac{1}{2}g_{uv}R = T_{uv} - \Lambda g_{uv} \tag{1.9}$$

where R_{uv}, R, T_{uv}, and Λ are the Ricci tensor, Ricci scalar, energy-momentum tensor, and cosmological constant, respectively, give the two independent equations

$$3\dot{\omega}^2 = \tfrac{1}{2}\dot{\phi}^2 + V(\phi) + \Lambda \tag{1.10}$$

$$2\ddot{\omega} + 3\dot{\omega}^2 = -\tfrac{1}{2}\dot{\phi}^2 + V(\phi) + \Lambda \tag{1.11}$$

where $\omega(t)$ is defined by $R(t) = e^{\omega(t)}$.

The Bianchi identities $T^{uv}{}_{;v} = 0$ are trivially satisfied since ϕ satisfies (1.4).

In other words, not all of equations (1.6), (1.10), and (1.11) are independent, which can be seen explicitly upon substitution of (1.10) in (1.11) to get

$$\ddot{\omega} = -\tfrac{1}{2}\dot{\phi}^2 \tag{1.12}$$

Now multiplication of (1.6) by $\dot{\phi}$ with the use of (1.12) results in

$$\dot{\phi}\ddot{\phi} + \dot{V}\dot{\phi} = 6\dot{\omega}\ddot{\omega} \tag{1.13}$$

which integrates immediately to become (1.10), where the constant can be taken to be Λ.

2. ϕ^4 FIELD

Suppose (Linde, 1979)

$$V(\phi)=\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}\lambda\phi^4 \tag{2.1}$$

The Einstein equations are

$$3\dot{\omega}^2=\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}\lambda\phi^4+\Lambda \tag{2.2}$$

$$2\ddot{\omega}+3\dot{\omega}^2=-\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}\lambda\phi^4+\Lambda \tag{2.3}$$

It can now be shown that (2.2) and (2.3) admit a solution for ϕ of the form $\phi=k\,e^{pt}$ with k and p real constants; using this form of ϕ in (1.12) yields

$$\dot{\omega}=-\tfrac{1}{4}(k^2p\,e^{2pt}-c),\qquad c=\text{const} \tag{2.4}$$

Substituting in (2.2) and (2.4) and equating powers of e^{pt} shows that the solution must have

$$\Lambda=\tfrac{3}{16}c^2,\qquad p^2+m^2=-\tfrac{3}{4}pc,\qquad \lambda=\tfrac{3}{4}p^2 \tag{2.5}$$

Suppose that we now choose $\Lambda>0$ and $\lambda>0$ and take $p>0$ and $c>0$, so that $p=2(\lambda/3)^{1/2}$. Then, writing $k=\phi_0(t=0)$, the solution is written as

$$\phi(t)=\phi_0\exp[2(\lambda/3)^{1/2}t] \tag{2.6}$$

$$R(t)=R_0\exp\{(\phi_0^2/8)\{1-\exp 4[(\lambda/3)^{1/2}t]\}+(\Lambda/3)^{1/2}t\} \tag{2.7}$$

where the constant obtained in the course of integrating equation (2.4) has been chosen so that

$$R(t=0)=R_0$$

Equations (2.5) show that $\text{Sgn}\{m^2\}=-\text{Sgn}\{\lambda\}$, so that the field is in a state of broken symmetry, with the minima of the potential energy $V(\phi)=\tfrac{1}{2}m^2+\tfrac{1}{4}\lambda\phi^4$ located at $\phi=\pm(u/\sqrt{\lambda})$, where $u^2=-m^2$ (Linde, 1979).

One thus expects that asymptotically $\phi\to\pm(u/\sqrt{\lambda})$. Examination of the solutions (2.6) and (2.7) shows that this is not the case. In fact, as $t\to-\infty$, $\phi\to 0$ and $R\to 0$ (the solution is thus asymptotically static). Also, both ϕ and R are finite as $t\to 0$. There is a singularity as $t\to\infty$, $\phi\to\infty$, and $R\to 0$.

We may also take $p<0$ and $c<0$, so that $p=-2(\lambda/3)^{1/2}$. The solution may be written as

$$\phi(t)=\phi_0\exp[-2(\lambda/3)^{1/2}t] \tag{2.8}$$

$$R(t)=R_0\exp\{(\phi_0^2/8)\{1-\exp[-4(\lambda/3)^{1/2}t]\}-(\Lambda/3)^{1/2}t\} \tag{2.9}$$

Equation (2.5) shows that $\text{Sgn}\{m^2\}=-\text{Sgn}\{\lambda\}$.

It can be shown that as $t \to \infty$, ϕ and $R \to 0$ (the solution is thus asymptotically static). Also, both ϕ and R are finite as $t \to 0$. There is a singularity as $t \to -\infty$, $\phi \to \infty$, and $R \to 0$.

Suppose that we now choose $\Lambda = 0$, $\lambda > 0$, and take $p > 0$ or $p < 0$, so that $p = \pm 2(\lambda/3)^{1/2}$. The solution may be written

$$\phi(t) = \phi_0 \exp[\pm 2(\lambda/3)^{1/2} t] \tag{2.10}$$

$$R(t) = R_0 \exp\{(\phi_0^2/8)\{1 - \exp[\pm 4(\lambda/3)^{1/2} t]\}\} \tag{2.11}$$

It can be shown that as $t \to \infty$, $\phi \to \infty$ (0) and $R \to 0$ ($R_0 \exp \phi_0^2/8$); the solution is thus singular (asymptotically static). Also, both ϕ and R are finite as $t \to 0$. The solution is asymptotically static (singular) as $t \to -\infty$, $\phi \to 0$ (∞), and $R \to R_0 \exp \phi_0^2/8$ (0).

Conclusion: The energy density on the right-hand side of equation (2.2) becomes infinite there.

3. ϕ^6 FIELD

Suppose

$$V(\phi) = \tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{4} \lambda \phi^4 + \tfrac{1}{6} l \phi^6 \tag{3.1}$$

The Einstein equations are

$$3\dot{\omega}^2 = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{4}\lambda\phi^4 + \tfrac{1}{6} l \phi^6 + \Lambda \tag{3.2}$$

$$2\ddot{\omega} + 3\dot{\omega}^2 = -\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{4}\lambda\phi^4 + \tfrac{1}{6} l \phi^6 + \Lambda \tag{3.3}$$

It can now be shown that (3.2) and (3.3) admit a solution for ϕ of the form $\phi = k$ ctgh pt. With k and p real constants, using this form of ϕ in (1.12) yields

$$\begin{aligned} \dot{\omega} &= \tfrac{1}{2} k^2 p (\tfrac{1}{3} \operatorname{ctgh}^3 pt - \operatorname{ctgh} pt) + c \\ c &= \text{const} \end{aligned} \tag{3.4}$$

Substituting in (3.2) and (3.4) and equating powers of ctgh pt shows that the solutions must have

$$c = 0, \qquad l = \tfrac{1}{2} p^2/k^2, \qquad \Lambda = -\tfrac{1}{2} k^2 p^2, \qquad m^2 = -k^2(\lambda + \tfrac{1}{2} p) \tag{3.5}$$

Suppose that we now choose $l > 0$ and $\Lambda < 0$ and take $p > 0$ or $p < 0$, so that $p = \pm 2^{1/2}(-l\Lambda)^{1/4}$. Then, writing $k = \phi_0 = \phi(t \to \pm\infty)$, we can write the solution as

$$\phi(t) = \phi_0 \operatorname{ctgh}[\pm 2^{1/2}(-l\Lambda)^{1/4} t] \tag{3.6}$$

$$\begin{aligned} R(t) = R_0 \exp\{-\tfrac{1}{3}k^2\{\ln \sinh[\pm 2^{1/2}(-l\Lambda)^{1/4} t] \\ + \tfrac{1}{4} \operatorname{ctgh}^2[\pm 2^{1/2}(-l\Lambda)^{1/4} t]\} + c_1\} \end{aligned} \tag{3.7}$$

where the constant obtained in the course of integrating equation (3.4) has been chosen so that

$$R(t = t_0) = R_0, \qquad c_1 = \frac{k^2}{3} (\ln \sinh pt_0 + \tfrac{1}{4} \operatorname{ctgh}^2 pt_0)$$

Equations (3.5) show that $\operatorname{Sgn}\{m^2\} = -\operatorname{Sgn}\{\lambda + l\}$. Thus, the field is in a state of broken symmetry with the minima of the potential energy

$$V(\phi) = \tfrac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \tfrac{1}{6}l\phi^6$$

located at

$$\phi = \pm\left[-\frac{\lambda}{2l} + \left(\frac{\lambda^2}{4l^2} + \frac{u^2}{l}\right)^{1/2}\right]^{1/2}$$

where $u^2 = -m^2$. One thus expects that asymptotically

$$\phi \to \pm\left[-\frac{\lambda}{2l} + \left(\frac{\lambda^2}{4l^2} + \frac{u^2}{l}\right)^{1/2}\right]^{1/2}$$

Examination of the solutions (3.6) and (3.7) shows that this is not the case. In fact, as $t \to +\infty$, $\phi \to \phi_0$ $(-\phi_0)$ and $R \to 0$ (singularity). As $t \to -\infty$, $\phi \to -\phi_0$ (ϕ_0) and $R \to$ singularity (0). As $t \to 0$, $\phi \to \infty$ and $R \to 0$. Hence, the energy density on the right-hand side of equation (3.2) becomes infinite there.

4. SINE-GORDON FIELD

Suppose (Rajarreman, 1975)

$$V(\phi) = \frac{\alpha}{\beta}(1 - \cos \beta\phi) \tag{4.1}$$

The Einstein equations are

$$3\dot{\omega}^2=\frac{1}{2}\dot{\phi}^2+\frac{\alpha}{\beta}(1-\cos\beta\phi)+\Lambda \tag{4.2}$$

$$2\ddot{\omega}+3\dot{\omega}^2=-\frac{1}{2}\dot{\phi}^2+\frac{\alpha}{\beta}(1-\cos\beta\phi)+\Lambda \tag{4.3}$$

It can now be shown that (4.2) and (4.3) admit a solution for ϕ of the form

$$\phi=\frac{2}{\beta}(\cos^{-1}\operatorname{tgh} pt+\pi k)$$

with k and p real constants. Using this form of ϕ in (1.12) yields

$$\dot{\omega}=-\frac{2p}{\beta^2}\operatorname{tgh} pt+c, \qquad c=\text{const} \tag{4.4}$$

Substituting in (4.2) and (4.3) and equating powers of tgh pt shows that the solution must have

$$c=0, \qquad \Lambda=\frac{12p^2}{\beta^4}, \qquad p^2+\alpha\beta=-\frac{6p^2}{\beta^2} \tag{4.5}$$

Comparing the sine-Gordon field with the ϕ^4 field, we easily obtain $m^2=\alpha\beta$. Suppose that we now choose $\Lambda>0$ and take $p>0$ or $p<0$, so that $p=\pm\frac{1}{2}\beta^2(\Lambda/3)^{1/2}$. Then by writing $\pi/\beta=\phi_0=\phi(t=0)$, we can write the solution as

$$\phi(t)=\frac{2}{\beta}\left\{\cos^{-1}\operatorname{tgh}\left[\pm\frac{\beta^2}{2}\left(\frac{\Lambda}{3}\right)^{1/2}t\right]+k\pi\right\} \tag{4.6}$$

$$R(t)=R_0\exp\left\{-\frac{2}{\beta^2}\ln\cosh\left[\pm\frac{\beta^2}{2}\left(\frac{\Lambda}{3}\right)^{1/2}t\right]\right\} \tag{4.7}$$

where the constant obtained in the course of integrating equation (4.4) has been chosen so that $R(t\to 0)=R_0$. Equation (4.5) shows that $\operatorname{Sgn}\{m^2\}=-\operatorname{Sgn}\{p^2\}$, so that the field is a state of broken symmetry with the minima of the potential energy $V(\phi)=(\alpha/\beta)(1-\cos\beta\phi)$ located at $\phi=\pm\pi/\beta$. One thus expects that asymptotically $\phi=\pm\pi/\beta$. Examination of the solutions (4.7) and (4.6) shows that this is not the case. In fact, as $t\to\infty$, $\phi\to 2k\pi/\beta[(2\pi/\beta)(k+1)]$ and $R\to 0$ (0). Also both ϕ and R are finite as $t\to 0$. As $t\to-\infty$, $\phi\to 2\pi/\beta(k+1)$ $(2k\pi/\beta)$ and $R\to 0$ (0). Hence, the energy density on the right-hand side of equation (4.2) becomes infinite as $k\to\infty$.

5. CONCLUSIONS

The Lagrangians in equation (1.1) are often encountered with positive-definite mass terms in discussions of symmetry breaking. Although the solution given here can be assumed to be extremely typical of cosmologies containing broken-symmetric matter fields, it is possible that the existence of the above solutions may have consequences for discussions of symmetry breaking in a cosmological context.

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